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On the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ and free boson representations

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Abstract
A two-parameter quantum deformation of the affine Lie superalgebra $osp(2|2)^{(2)}$ is introduced and studied in some detail. This algebra is the first example associated with nonsimply laced and twisted root systems of a quantum current algebra with the structure of a so-called infinite Hopf family of (super)algebras. A representation of this algebra at $c = 1$ is realized in the product Fock space of two commuting sets of Heisenberg algebras.

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1. Introduction

The study of two-parameter deformations of affine Lie (super)algebras has turned out to be quite fruitful in the last few years. These algebras are, in a sense, deformations of the standard quantum affine algebras or Yangian doubles, which all have the structure of quasi-triangular Hopf algebras. In contrast, the two-parameter deformations are not Hopf algebras, but rather have twisted Hopf structures, one of which is the Drinfeld twist of Hopf algebras or quasi-Hopf algebras, while the other is the infinite Hopf family of (super)algebras. So far, the relationship between the two generalized co-structures remains ill-understood.

From the physical point of view, the above two classes of two-parameter deformations appear in different contexts. Quasi-Hopf algebras [1] occur in the study of symmetries of face- and vertex-type models of statistical mechanics [6], and are closely related to the face-type Boltzmann weights (Yang–Baxter R matrix), while infinite Hopf families of (super)algebras [4, 5] occur only in the representation theory of quantum deformed Virasoro and W algebras [2] (and act as the algebra of screening currents [3, 4]), which in turn are algebras characterizing the dynamical symmetries of certain massive integrable quantum field theories [7].

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Despite the great deal of work that has been done on these two-parameter deformations, many problems remain unsolved. In particular, for the second class of two-parameter deformations (the infinite Hopf families) nothing has been said concerning root systems of nonsimply laced and/or twisted types. For super root systems, the only case which has been considered is the case of $osp(1|2)^{(1)}$ [10]. In this paper, therefore, we aim to provide more concrete examples of this kind, and in particular the trigonometric two-parameter deformation of $osp(2|2)^{(2)}$. This algebra is based on a root system which is simultaneously non-simply laced, twisted and super. From the known relationship between two-parameter deformations of other affine Lie (super)algebras we expect that the algebra we study in this paper should be interesting in the same way: it should correspond to the algebra of screening currents of quantum deformed $N = 2$ superconformal algebra (if the latter exists).

2. Definition and structure of the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$

2.1. Definition of $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$

We start with the definition of the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$. The notation follows that of $\mathcal{A}_{\hbar,\eta}(\hat{g})$ [4, 5, 7] and $\mathcal{A}_{\hbar,\eta}(osp(1|2)^{(1)})$ [10].

Definition 2.1. *The algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$, considered as a continuously distributed current (super)algebra, is a Z_2 graded associative algebra over C generated by the currents $E(u)$, $F(u)$, $H^{\pm}(u)$, the central element c and unit 1 with parities $\pi[E(u)] = \pi[F(u)] = 1$, $\pi[H^{\pm}(u)] = \pi[c] = \pi[1] = 0$ and generating elations*

$$E(u)E(v) = -\frac{\cosh[\pi\eta(u-v+\mathrm{i}\hbar)]}{\cosh[\pi\eta(u-v-\mathrm{i}\hbar)]}E(v)E(u) \tag{1}$$

$$F(u)F(v) = -\frac{\cosh[\pi\eta'(u-v-\mathrm{i}\hbar)]}{\cosh[\pi\eta'(u-v+\mathrm{i}\hbar)]}F(v)F(u) \tag{2}$$

$$H^{\pm}(u)E(v) = \frac{\cosh[\pi\eta(u-v+\mathrm{i}\hbar\pm\mathrm{i}\hbar c/4)]}{\cosh[\pi\eta(u-v-\mathrm{i}\hbar\pm\mathrm{i}\hbar c/4)]}E(v)H^{\pm}(u) \tag{3}$$

$$H^{\pm}(u)F(v) = \frac{\cosh[\pi\eta'(u-v-\mathrm{i}\hbar\mp\mathrm{i}\hbar c/4)]}{\cosh[\pi\eta'(u-v+\mathrm{i}\hbar\mp\mathrm{i}\hbar c/4)]}F(v)H^{\pm}(u) \tag{4}$$

$$H^{\pm}(u)H^{\pm}(v) = \frac{\cosh[\pi\eta(u-v+\mathrm{i}\hbar)]}{\cosh[\pi\eta(u-v-\mathrm{i}\hbar)]}\frac{\cosh[\pi\eta'(u-v-\mathrm{i}\hbar)]}{\cosh[\pi\eta'(u-v+\mathrm{i}\hbar)]}H^{\pm}(v)H^{\pm}(u) \tag{5}$$

$$H^{+}(u)H^{-}(v) = \frac{\cosh[\pi\eta(u-v+\mathrm{i}\hbar+\mathrm{i}\hbar c/2)]}{\cosh[\pi\eta(u-v-\mathrm{i}\hbar+\mathrm{i}\hbar c/2)]}\frac{\cosh[\pi\eta'(u-v-\mathrm{i}\hbar-\mathrm{i}\hbar c/2)]}{\cosh[\pi\eta'(u-v+\mathrm{i}\hbar-\mathrm{i}\hbar c/2)]}H^{-}(v)H^{+}(u) \tag{6}$$

$$\{E(u), F(v)\} = \frac{2\pi}{\hbar}\left[\delta(u-v-\mathrm{i}\hbar c/2)H^{+}(u-\mathrm{i}\hbar c/4) - \delta(u-v+\mathrm{i}\hbar c/2)H^{-}(v-\mathrm{i}\hbar c/4)\right] \tag{7}$$

where

$$\frac{1}{\eta'} - \frac{1}{\eta} = \hbar c$$

and $\hbar$ and η are generic deformation parameters.

For later reference, we denote the subalgebras generated respectively by the currents $\{E(u)\}$ and $\{F(u)\}$ as $\mathcal{N}_{\pm}[\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})]$.

It is interesting to compare the generating relations of the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ and those of the quantum affine (super)algebra $U_q(osp(2|2)^{(2)})$. The latter has the generating relations [9]

$$X^+(z)X^+(w) = -\frac{zq+w}{z+wq}X^+(w)X^+(z) \tag{8}$$

$$X^-(z)X^-(w) = -\frac{z+wq}{zq+w}X^-(w)X^-(w) \tag{9}$$

$$\psi^\pm(z)X^+(w) = \frac{z_\pm q+w}{z_\pm+wq}X^+(w)\psi^\pm(z) \tag{10}$$

$$\psi^\pm(z)X^-(w) = \frac{z_\mp+wq}{z_\mp q+w}X^-(w)\psi^\pm(z) \tag{11}$$

$$\psi^\pm(z)\psi^\pm(w) = \psi^\pm(w)\psi^\pm(z) \tag{12}$$

$$\psi^+(z)\psi^-(w) = \frac{z_+q+w_-}{z_++w_-q}\frac{z_-+w_+q}{z_-q+w_+}\psi^-(w)\psi^+(z) \tag{13}$$

$$\{X^+(z), X^-(w)\} = \frac{1}{(q-q^{-1})zw}\left[\delta\left(\frac{w}{z}\gamma\right)\psi^+(z_-) - \delta\left(\frac{w}{z}\gamma^{-1}\right)\psi^-(w_-)\right] \tag{14}$$

where[4] $z_\pm = z\gamma^{\pm 1/2}$. Notice that the δ functions appearing in (7) and (14) are supported differently, the former at 0 (i.e. the standard Dirac δ function) and the latter at 1 ($\delta(z) \equiv \sum_{n\in Z} z^n$ as a formal power series). We also denote the subalgebras of $U_q(osp(2|2)^{(2)})$ generated respectively by $X^+(z)$ and $X^-(z)$ by $\mathcal{N}_\pm[U_q(osp(2|2)^{(2)})]$.

The following two propositions justify the similarities between the two algebras $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ and $U_q(osp(2|2)^{(2)})$: first,

Proposition 2.2. *There are algebra homomorphisms* $\rho^+ : \mathcal{N}_+[\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})] \to \mathcal{N}_+[U_q(osp(2|2)^{(2)})]$, $\rho^- : \mathcal{N}_-[\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})] \to \mathcal{N}_-[U_{q'}(osp(2|2)^{(2)})]$, *where under* $\rho^\pm$ *the parameters behave as*

$$\rho^+(\mathrm{e}^{2\pi\eta u}) = z$$
$$\rho^+(\mathrm{e}^{2\pi\mathrm{i}\eta\hbar}) = q$$

and

$$\rho^-(\mathrm{e}^{2\pi\eta' u}) = z$$
$$\rho^-(\mathrm{e}^{2\pi\mathrm{i}\eta'\hbar}) = q'$$

respectively.

Recalling that η and η' are different only when $c \neq 0$, we also have

Proposition 2.3. *There is an algebra homomorphism between* $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ *at* $c = 0$ *and* $U_q(osp(2|2)^{(2)})$ *at* $\gamma = 1$*:*

$$\mathcal{E} : \mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)}) \to U_q(osp(2|2)^{(2)})$$
$$E(u) \longmapsto \sqrt{2}zX^+(z)$$
$$F(u) \longmapsto \sqrt{2}zX^-(z)$$
$$\frac{2\pi}{\hbar}H^\pm(u) \longmapsto \frac{1}{q-q^{-1}}\psi^\pm(z)$$

where $z = \mathrm{e}^{2\pi\eta\hbar u}$, $q = \mathrm{e}^{2\pi\mathrm{i}\eta\hbar}$.

[4] In the original presentation of $U_q(osp(2|2)^{(2)})$ in [9], the element γ was written as q^c. However, to avoid confusion with the central element c of the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$, we intentionally rename it γ, as is usual in ordinary quantum affine algebras.

Proposition 2.2 indicates that the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ is actually an interpolation between (Borel subalgebras of) two standard quantum affine algebras $U_q(osp(2|2)^{(2)})$ and $U_{q'}(osp(2|2)^{(2)})$ with different deformation parameters, while proposition 2.3 further states that, at $c = 0$, the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ degenerates into $U_q(osp(2|2)^{(2)})$ at $\gamma = 1$.

2.2. Co-structure

As expected, this algebra possesses the structure of an infinite Hopf family of (super)algebras, whose definition can be found in [10] (see also [4, 5]). In fact, if we denote $\mathcal{A}_n = \mathcal{A}_{\hbar,\eta^{(n)}}(osp(2|2)^{(2)})_{c_n}$, where $\eta^{(n)}$ is defined iteratively via $\frac{1}{\eta^{(n+1)}} - \frac{1}{\eta^{(n)}} = \hbar c_n$ starting from $\eta^{(1)} = \eta$ and taking $c_n \in Z\backslash Z_-$, we can define the following co-structures over the family of algebras $\{\mathcal{A}_n, n \in Z\}$:

- the co-multiplications $\Delta_n^{\pm}$ (algebra homomorphisms $\Delta_n^+ : \mathcal{A}_n \to \mathcal{A}_n \otimes \mathcal{A}_{n+1}$, $\Delta_n^- : \mathcal{A}_n \to \mathcal{A}_{n-1} \otimes \mathcal{A}_n$):

$$\Delta_n^+ c_n = c_n + c_{n+1}$$
$$\Delta_n^- c_n = c_{n-1} + c_n$$
$$\Delta_n^+ H^+(u;\eta^{(n)}) = H^+\left(u + \frac{\mathrm{i}\hbar c_{n+1}}{4};\eta^{(n)}\right) \otimes H^+\left(u - \frac{\mathrm{i}\hbar c_n}{4};\eta^{(n+1)}\right)$$
$$\Delta_n^- H^+(u;\eta^{(n)}) = H^+\left(u + \frac{\mathrm{i}\hbar c_n}{4};\eta^{(n-1)}\right) \otimes H^+\left(u - \frac{\mathrm{i}\hbar c_{n-1}}{4};\eta^{(n)}\right)$$
$$\Delta_n^+ H^-(u;\eta^{(n)}) = H^-\left(u - \frac{\mathrm{i}\hbar c_{n+1}}{4};\eta^{(n)}\right) \otimes H^-\left(u + \frac{\mathrm{i}\hbar c_n}{4};\eta^{(n+1)}\right)$$
$$\Delta_n^- H^-(u;\eta^{(n)}) = H^-\left(u - \frac{\mathrm{i}\hbar c_n}{4};\eta^{(n-1)}\right) \otimes H^-\left(u + \frac{\mathrm{i}\hbar c_{n-1}}{4};\eta^{(n)}\right)$$
$$\Delta_n^+ E(u;\eta^{(n)}) = E(u;\eta^{(n)}) \otimes 1 + H^-\left(u + \frac{\mathrm{i}\hbar c_n}{4};\eta^{(n)}\right) \otimes E\left(u + \frac{\mathrm{i}\hbar c_n}{2};\eta^{(n+1)}\right)$$
$$\Delta_n^- E(u;\eta^{(n)}) = E(u;\eta^{(n-1)}) \otimes 1 + H^-\left(u + \frac{\mathrm{i}\hbar c_{n-1}}{4};\eta^{(n-1)}\right) \otimes E\left(u + \frac{\mathrm{i}\hbar c_{n-1}}{2};\eta^{(n)}\right)$$
$$\Delta_n^+ F(u;\eta^{(n)}) = 1 \otimes F(u;\eta^{(n+1)}) + F\left(u + \frac{\mathrm{i}\hbar c_{n+1}}{2};\eta^{(n)}\right) \otimes H^+\left(u + \frac{\mathrm{i}\hbar c_{n+1}}{4};\eta^{(n+1)}\right)$$
$$\Delta_n^- F(u;\eta^{(n)}) = 1 \otimes F(u;\eta^{(n)}) + F\left(u + \frac{\mathrm{i}\hbar c_n}{2};\eta^{(n-1)}\right) \otimes H^+\left(u + \frac{\mathrm{i}\hbar c_n}{4};\eta^{(n)}\right)$$

- the co-units ϵ_n (algebra homomorphism $\epsilon_n : \mathcal{A}_n \to C$):

$$\epsilon_n(c_n) = 0$$
$$\epsilon_n(1_n) = 1$$
$$\epsilon_n(H_i^{\pm}(u;\eta^{(n)})) = 1$$
$$\epsilon_n(E_i(u;\eta^{(n)})) = 0$$
$$\epsilon_n(F_i(u;\eta^{(n)})) = 0$$

- the antipodes $S_n^{\pm}$ (algebra anti-homomorphisms $S_n^{\pm} : \mathcal{A}_n \to \mathcal{A}_{n\pm1}$):

$$S_n^{\pm}(c_n) = -c_{n\pm1}$$
$$S_n^{\pm}(H^{\pm}(u;\eta^{(n)})) = [H^{\pm}(u;\eta^{(n\pm1)})]^{-1}$$
$$S_n^{\pm}(E(u;\eta^{(n)})) = -H^-\left(u - \frac{\mathrm{i}\hbar c_{n\pm1}}{4};\eta^{(n\pm1)}\right)^{-1} E\left(u - \frac{\mathrm{i}\hbar c_{n\pm1}}{2};\eta^{(n\pm1)}\right)$$

$$S_n^{\pm}(F(u;\eta^{(n)})) = -F\left(u - \frac{\mathrm{i}\hbar c_{n\pm1}}{2};\eta^{(n\pm1)}\right) H^{+}\left(u - \frac{\mathrm{i}\hbar c_{n\pm1}}{4};\eta^{(n\pm1)}\right)^{-1}$$

where $\otimes$ stands for the direct super product defined by

$$(A \otimes B)(C \otimes D) = (-1)^{\pi[B]\pi[C]} AB \otimes CD$$

for homogeneous elements A, B, C, D. It is a trivial (but tedious) exercise to check that these structures satisfy all the defining axioms of an infinite Hopf family of (super)algebras,

- $(\epsilon_n \otimes id_{n+1}) \circ \Delta_n^+ = \tau_n^+,\ (id_{n-1} \otimes \epsilon_n) \circ \Delta_n^- = \tau_n^-$
- $m_{n+1} \circ (S_n^+ \otimes id_{n+1}) \circ \Delta_n^+ = \epsilon_{n+1} \circ \tau_n^+,\ m_{n-1} \circ (id_{n-1} \otimes S_n^-) \circ \Delta_n^- = \epsilon_{n-1} \circ \tau_n^-$
- $(\Delta_n^- \otimes id_{n+1}) \circ \Delta_n^+ = (id_{n-1} \otimes \Delta_n^+) \circ \Delta_n^-$

where m_n is the (super)multiplication for $\mathcal{A}_n$ and $\tau_n^{\pm}$ are algebra shift morphisms $\tau_n^{\pm} : \mathcal{A}_n \to \mathcal{A}_{n\pm1}$ which obey $\tau_{n+1}^- \tau_n^+ = id_n = \tau_{n-1}^+ \tau_n^-$.

The operations $\Delta_n^{\pm}$ are related to each other by the shift morphisms:

$$\Delta_n^- = (\tau_n^- \otimes \tau_{n+1}^-) \circ \Delta_n^+$$
$$\Delta_n^+ = (\tau_{n-1}^+ \otimes \tau_n^+) \circ \Delta_n^-.$$

Thus the easily observed co-commutativity between the two co-multiplications

$$(\Delta_n^- \otimes id_{n+1}) \circ \Delta_n^+ = (id_{n-1} \otimes \Delta_n^+) \circ \Delta_n^-$$

can be rewritten in terms of only one of the two co-multiplications, and turns out to become a statement of the non-co-associativity of the co-multiplications:

$$[((\tau_n^- \otimes \tau_{n+1}^-) \circ \Delta_n^+) \otimes id_{n+1}] \circ \Delta_n^+ = (id_{n-1} \otimes \Delta_n^+) \circ ((\tau_n^- \otimes \tau_{n+1}^-) \circ \Delta_n^+)$$
$$(\Delta_n^- \otimes id_{n+1}) \circ ((\tau_{n-1}^+ \otimes \tau_n^+) \circ \Delta_n^-) = [id_{n-1} \otimes ((\tau_{n-1}^+ \otimes \tau_n^+) \circ \Delta_n^-)] \circ \Delta_n^-.$$

Notice that these twisted co-associativity conditions are different from that of the Drinfeld twists. However, the effects of these two different kinds of twists are the same: they all allow one to construct fused (tensor product) representations for the algebras under investigation, although the co-structures are not co-associative.

Now recall that definition 2.1 defines the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ only as a formal algebra, in the sense that all currents thus defined are actually only distributions. To assign precise meaning to the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ we need to specify the actual generators and relations, and this can be done only separately for two distinct cases $c = 0$ and $c \neq 0$ (as in the case of $\mathcal{A}_{\hbar,\eta}(\widehat{sl_2})$ [7] and $\mathcal{A}_{\hbar,\eta}(\hat{g})$ [4]). For details, the reader is directed to Khoroshkin *et al* [7] in the $\widehat{sl}_2$ case. The present case is in complete analogy.

3. Representation theory

3.1. Case $c = 0$

Recall that, for $c = 0$, there is an algebra homomorphism between the algebras $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ and $U_q(osp(2|2)^{(2)})$ for $q = \mathrm{e}^{2\pi\mathrm{i}\eta\hbar}$. Thus the evaluation representation of $U_q(osp(2|2)^{(2)})$ presented in [9] can be extended into an evaluation representation of $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ in terms of the evaluation homomorphism $\mathcal{E}$. This evaluation representation justifies the relationship between the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ and the root system of type $osp(2|2)^{(2)}$.

3.2. Case $c = 1$ and structure of the Fock space

As usual, the tool we need to construct a representation of $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ at $c = 1$ is the free boson realization. Throughout this subsection we have $1/\eta' = 1/\eta + \hbar$.

Define the Heisenberg algebras $\mathcal{H}_\alpha, \mathcal{H}_\beta$ respectively by

$$[\alpha(\lambda), \alpha(\mu)] = A(\lambda)\delta(\lambda + \mu)$$
$$[\beta(\lambda), \beta(\mu)] = B(\lambda)\delta(\lambda + \mu)$$
$$[\alpha(\lambda), \beta(\mu)] = 0 \qquad (\lambda \neq \mu)$$

where $A(\lambda)$ and $B(\lambda)$ are given as

$$A(\lambda) = \frac{\lambda}{4\cosh\frac{\hbar\lambda}{2} + (\operatorname{csch}\frac{\lambda}{2\eta} - \operatorname{csch}\frac{\lambda}{2\eta'})\sinh\hbar\lambda + 2}$$
$$B(\lambda) = \frac{\lambda((1 + \operatorname{csch}\frac{\lambda}{2\eta}\sinh\hbar\lambda)(1 - \operatorname{csch}\frac{\lambda}{2\eta'}\sinh\hbar\lambda) - 4\cosh^2\frac{\hbar\lambda}{2})}{4\cosh\frac{\hbar\lambda}{2} + (\operatorname{csch}\frac{\lambda}{2\eta} - \operatorname{csch}\frac{\lambda}{2\eta'})\sinh\hbar\lambda + 2}$$

both of which are antisymmetric as $\lambda \to -\lambda$ and regular as $\lambda \to 0$. In fact, we can easily check that

$$A(\lambda) = -A(-\lambda)$$
$$B(\lambda) = -B(-\lambda)$$

and

$$A(\lambda) \sim \frac{\lambda}{2(\eta - \eta')\hbar + 6} + O(\lambda^3)$$
$$B(\lambda) \sim \frac{[(1 + 2\hbar\eta)(1 - 2\hbar\eta') - 4]\lambda}{2(\eta - \eta')\hbar + 6} + O(\lambda^3)$$

indicating that the Heisenberg algebras $\mathcal{H}_\alpha, \mathcal{H}_\beta$ are well defined even at $\lambda = 0$. The conjugates of $\alpha(0)$ and $\beta(0)$ have to be introduced separately, however, as follows. Let $Q_\alpha = \alpha(0)$, $Q_\beta = \beta(0)$ and their conjugate operators P_α, P_β be defined by the following relations:

$$[P_\alpha, Q_\alpha] = 1$$
$$[P_\beta, Q_\beta] = 1$$
$$[P_\alpha, Q_\beta] = [P_\beta, Q_\alpha] = 0.$$

Now denoting

$$X_a(\lambda) = \frac{1}{\hbar\lambda}\left(\operatorname{csch}\frac{\lambda}{2\eta}\sinh\hbar\lambda + 2\cosh\frac{\hbar\lambda}{2} + 1\right)$$
$$X_b(\lambda) = \frac{1}{\hbar\lambda}\left(\operatorname{csch}\frac{\lambda}{2\eta'}\sinh\hbar\lambda - 2\cosh\frac{\hbar\lambda}{2} - 1\right)$$
$$Y_a(\lambda) = Y_b(\lambda) = \frac{1}{\hbar\lambda}$$

we can define

$$a(\lambda) = X_a(\lambda)\alpha(\lambda) + Y_a(\lambda)\beta(\lambda)$$
$$b(\lambda) = X_b(\lambda)\alpha(\lambda) + Y_b(\lambda)\beta(\lambda) \qquad (\lambda \neq \mu)$$

so that the corresponding commutation relations are

$$[a(\lambda), a(\mu)] = -\frac{1}{\hbar^2\lambda}\left(1 + \frac{\sinh\hbar\lambda}{\sinh\frac{\lambda}{2\eta}}\right)\delta(\lambda + \mu) \tag{15}$$

$$[b(\lambda), b(\mu)] = -\frac{1}{\hbar^2\lambda}\left(1 - \frac{\sinh \hbar\lambda}{\sinh \frac{\lambda}{2\eta'}}\right)\delta(\lambda+\mu) \tag{16}$$

$$[a(\lambda), b(\mu)] = [b(\lambda), a(\mu)] = \frac{2}{\hbar^2\lambda}\cosh\frac{\hbar\lambda}{2}\delta(\lambda+\mu) \qquad (\lambda \neq \mu). \tag{17}$$

These commutation relations are crucial for the construction of the free boson representation for the algebra $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ and hence we give this algebra a short name for reference: $\mathcal{H}[a, b]$. Recall that we are dealing with *generic* deformation parameters; we do not consider the specific values of the parameters at which the above bosonic algebra becomes ill-defined (these include the points at which $\frac{1}{2\eta}$ is a rational multiple of $\hbar$).

Before going into the details of the representation theory, we have to specify the structure of the Fock space on which the free bosonic algebra $\mathcal{H}[a, b]$ acts. Actually, there are infinitely many ways to realize the bosonic algebra $\mathcal{H}[a, b]$ in terms of two commuting sets of Heisenberg algebras, and what we outlined above is only one of infinitely many choices.

Denoting respectively by $\mathcal{F}_\alpha$ and $\mathcal{F}_\beta$ the Fock spaces for the Heisenberg algebras $\mathcal{H}_\alpha$ and $\mathcal{H}_\beta$, we see that the bosonic algebra $\mathcal{H}[a, b]$ can be realized in a proper subspace $\mathcal{F}[a, b]$ of $\mathcal{F}_\alpha \otimes \mathcal{F}_\beta$ by actions of the form

$$\begin{aligned}
\mathcal{F}_\alpha \otimes \mathcal{F}_\beta \supset \mathcal{F}[a, b] \ni |v_{f_1,\ldots,f_n,g_1,\ldots,g_m}\rangle &= \int_{-\infty}^{-\epsilon} \mathrm{d}\lambda_1\, f(\lambda_1)a(\lambda_1)\cdots \int_{-\infty}^{-\epsilon} \mathrm{d}\lambda_n\, f(\lambda_n)a(\lambda_n) \\
&\quad\times \int_{-\infty}^{-\epsilon} \mathrm{d}\mu_1\, g(\mu_1)b(\mu_1)\cdots \int_{-\infty}^{-\epsilon} \mathrm{d}\mu_m\, g(\mu_m)b(\mu_m)|0\rangle_\alpha \otimes |0\rangle_\beta \\
&\quad \lambda_1,\ldots,\lambda_n; \mu_1,\ldots,\mu_m < 0 \quad \forall n, m \in Z \quad 0 < \epsilon \to 0^+.
\end{aligned}$$

Notice that the ordering of a and b in the above expression is irrelevant, because all values of $\lambda_1, \ldots, \lambda_n$ and $\mu_1, \ldots, \mu_m$ are negative. The Fock space thus described gives the left action (or action onto the right) of the algebra $\mathcal{H}[a, b]$. The Fock space which provides the right action (or action onto the left) can be specified as the conjugation of the above, i.e. $\mathcal{F}^*[a, b]$.

It remains to specify the correlation functions for operators acting on the Fock spaces $\mathcal{F}[a, b]$ and $\mathcal{F}^*[a, b]$ or, using more precise mathematical terminology, the pairing $\mathcal{F}[a, b] \otimes \mathcal{F}^*[a, b] \to C$. This is given by the following three steps. First, we fix the normalization for the vacuum vectors as follows:

$$({}_\alpha\langle 0| \otimes_\beta \langle 0|)(|0\rangle_\alpha \otimes |0\rangle_\beta) = 1.$$

Next, for any two vectors

$$\langle v_{f_i}| =_\alpha \langle 0| \otimes_\beta \langle 0| \int_\epsilon^{+\infty} \mathrm{d}\lambda\, f_i(\lambda)X_i(\lambda)$$

$$|v_{g_j}\rangle = \int_{-\infty}^{-\epsilon} \mathrm{d}\mu\, g_j(\mu)X_j(\mu)|0\rangle_\alpha \otimes |0\rangle_\beta$$

where $X_{i,j}(\lambda)$ are operators acting on the Fock spaces $\mathcal{F}[a, b]$ and $\mathcal{F}^*[a, b]$ satisfying

$$X_{i,j}(\lambda)|0\rangle_\alpha \otimes |0\rangle_\beta = 0 =_\alpha \langle 0| \otimes_\beta \langle 0|X_{i,j}(-\lambda) \qquad (\lambda > 0)$$

$$[X_i(\lambda), X_j(\mu)] = x_{ij}(\lambda)\delta(\lambda+\mu) \qquad (x_{ij}(\lambda) \text{ regular at } \lambda = 0)$$

with $f_i(\lambda)$ and $g_j(\lambda)$ both analytic in a small neighbourhood of $\lambda = 0$, except at $\lambda = 0$ where they have simple poles, we define the inner product as follows:

$$\langle v_{f_i}|v_{g_j}\rangle = \int_C \frac{\mathrm{d}\lambda \ln(-\lambda)}{2\pi\mathrm{i}} f_i(\lambda)x_{ij}(\lambda)g_j(-\lambda)$$

where C is an integration contour which goes from infinity to zero above the positive real λ axis, surrounding the origin counterclockwise, and going to infinity again below the positive real λ axis. This particular kind of regularization has already been used in [4, 7]. Last, for 'multi-particle' states like $\langle v_{f_{i_1},\ldots,f_{i_k}}|$ and $|v_{g_{j_1},\ldots,g_{j_k}}\rangle$, we apply the Wick theorem.

Having provided all the necessary tools for defining the free boson representation, we now introduce the notation

$$\varphi(u) = \int_{\lambda\neq 0} \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}\lambda u} a(\lambda) \tag{18}$$

$$\phi(u) = \int_{\lambda\neq 0} \mathrm{d}\lambda\, \mathrm{e}^{\mathrm{i}\lambda u} b(\lambda) \tag{19}$$

where $\int_{\lambda\neq 0}\mathrm{d}\lambda$ means the integration over the whole real λ axis except the point $\lambda = 0$, i.e.

$$\int_{\lambda\neq 0} \mathrm{d}\lambda = \lim_{\epsilon\to 0^+}\left(\int_{-\infty}^{-\epsilon}\mathrm{d}\lambda + \int_{\epsilon}^{+\infty}\mathrm{d}\lambda\right).$$

We then have

Proposition 3.1. *The following expressions give a free boson realization of the algebra* $\mathcal{A}_{\hbar,\eta}(osp(2|2)^{(2)})$ *at* $c = 1$*:*

$$E(u) = \mathrm{e}^{\gamma_{\mathrm{E}}-\ln\eta} : \exp\left[\frac{\mathrm{i}\pi}{2}\left(\frac{1}{p_\alpha}P_\alpha + \frac{1}{p_\beta}P_\beta\right) + (p_\alpha Q_\alpha + p_\beta Q_\beta)\right]\exp\hbar[\varphi(u)] :$$

$$F(u) = \mathrm{e}^{\gamma_{\mathrm{E}}-\ln\eta} : \exp\left[\frac{\mathrm{i}\pi}{2}\left(\frac{1}{p_\alpha}P_\alpha - \frac{1}{p_\beta}P_\beta\right) + (p_\alpha Q_\alpha + p_\beta Q_\beta)\right]\exp\hbar[\phi(u)]$$

$$H^{\pm}(u) =: \exp\left[\mathrm{i}\pi\left(\frac{1}{p_\alpha}P_\alpha\right) + 2(p_\alpha Q_\alpha + p_\beta Q_\beta)\right]\exp\hbar[\varphi(u \pm \mathrm{i}\hbar/4) + \phi(u \mp \mathrm{i}\hbar/4)] :$$

where p_α*,* p_β *are two arbitrary nonzero constants and* γ_{E} *is the Euler constant* $\gamma_{\mathrm{E}} = 0.577\,215\,66\cdots$.

The proof is by straightforward calculation using the Fock space conventions above. The following formulae play crucial roles:

$$\int_C \frac{\mathrm{d}\lambda\,\ln(-\lambda)}{2\pi\mathrm{i}\lambda}\frac{\mathrm{e}^{-x\lambda}}{1-\mathrm{e}^{-\lambda/\eta}} = \ln\Gamma(\eta x) + \left(\eta x - \frac{1}{2}\right)(\gamma_{\mathrm{E}} - \ln\eta) - \frac{1}{2}\ln(2\pi)$$

$$\Gamma\left(\frac{1}{2} - x\right)\Gamma\left(\frac{1}{2} + x\right) = \frac{\pi}{\cos\pi x}.$$

3.3. Free boson representations of $U_q(osp(2|2)^{(2)})$

In this subsection we carry out the same procedure as in the last subsection without assuming the relation $1/\eta' = 1/\eta + \hbar$. As we shall see, this yields a representation of the algebra $U_q(osp(2|2)^{(2)})$ at $\gamma = q^{1/2}$. Below we give some of the details.

We introduce two Heisenberg algebras $\tilde{\mathcal{H}}_\alpha$, $\tilde{\mathcal{H}}_\beta$, not to be confused with those of the last subsection, defined respectively by

$$[\alpha(\lambda), \alpha(\mu)] = A(\lambda)\delta(\lambda + \mu)$$

$$[\beta(\lambda), \beta(\mu)] = B(\lambda)\delta(\lambda + \mu)$$

$$[\alpha(\lambda), \beta(\mu)] = 0 \qquad (\lambda \neq \mu)$$

where $A(\lambda)$ and $B(\lambda)$ are given by

$$A(\lambda) = \frac{\lambda}{4\cosh\frac{\hbar\lambda}{2} + 2}$$

$$B(\lambda) = -\frac{\lambda(\operatorname{csch}^2\frac{\lambda}{2\eta}\sinh^2\hbar\lambda + 2\cosh\hbar\lambda + 1)}{4\cosh\frac{\hbar\lambda}{2} + 2}.$$

The zero mode operators remain as in the last subsection:

$$[P_\alpha, Q_\alpha] = 1$$
$$[P_\beta, Q_\beta] = 1$$
$$[P_\alpha, Q_\beta] = [P_\beta, Q_\alpha] = 0.$$

Let

$$X_a(\lambda) = \frac{1}{\lambda}\left(\operatorname{csch}\frac{\lambda}{2\eta}\sinh\hbar\lambda + 2\cosh\frac{\hbar\lambda}{2} + 1\right)$$
$$X_b(\lambda) = \frac{1}{\lambda}\left(\operatorname{csch}\frac{\lambda}{2\eta}\sinh\hbar\lambda - 2\cosh\frac{\hbar\lambda}{2} - 1\right)$$
$$Y_a(\lambda) = Y_b(\lambda) = \frac{1}{\lambda}$$

and define

$$a(\lambda) = X_a(\lambda)\alpha(\lambda) + Y_a(\lambda)\beta(\lambda)$$
$$b(\lambda) = X_b(\lambda)\alpha(\lambda) + Y_b(\lambda)\beta(\lambda) \qquad (\lambda \neq \mu)$$

so that the corresponding commutation relations are

$$[a(\lambda), a(\mu)] = -\frac{1}{\lambda}\left(1 + \frac{\sinh\hbar\lambda}{\sinh\frac{\lambda}{2\eta}}\right)\delta(\lambda+\mu) \tag{20}$$

$$[b(\lambda), b(\mu)] = -\frac{1}{\lambda}\left(1 - \frac{\sinh\hbar\lambda}{\sinh\frac{\lambda}{2\eta}}\right)\delta(\lambda+\mu) \tag{21}$$

$$[a(\lambda), b(\mu)] = [b(\lambda), a(\mu)] = \frac{2}{\lambda}\cosh\frac{\hbar\lambda}{2}\delta(\lambda+\mu) \qquad (\lambda \neq \mu). \tag{22}$$

We introduce the free bosonic fields $\varphi(u)$ and $\phi(u)$ as in (19) but using the Heisenberg algebras described in this subsection.

Proposition 3.2. *The following expressions give a free boson realization of the algebra $U_q(osp(2|2)^{(2)})$ currents (8)–(14) at $\gamma = q^{1/2}$:*

$$X^+(z) = \mathrm{e}^{\gamma_{\mathrm{E}}-\ln\eta}\frac{1}{\sqrt{2}z} : \exp\left[\frac{\mathrm{i}\pi}{2}\left(\frac{1}{p_\alpha}P_\alpha + \frac{1}{p_\beta}P_\beta\right) + (p_\alpha Q_\alpha + p_\beta Q_\beta)\right]$$
$$\times \exp\left[\varphi\left(\frac{\ln z}{2\pi\eta}\right)\right] :$$
$$X^-(z) = \mathrm{e}^{\gamma_{\mathrm{E}}-\ln\eta}\frac{1}{\sqrt{2}z} : \exp\left[\frac{\mathrm{i}\pi}{2}\left(\frac{1}{p_\alpha}P_\alpha - \frac{1}{p_\beta}P_\beta\right) + (p_\alpha Q_\alpha + p_\beta Q_\beta)\right]$$
$$\times \exp\left[\phi\left(\frac{\ln z}{2\pi\eta}\right)\right] :$$
$$\psi^\pm(z) = \frac{(2\pi)^2\mathrm{i}\eta(q-q^{-1})}{\ln q} : \exp\left[\mathrm{i}\pi\left(\frac{1}{p_\alpha}P_\alpha\right) + 2(p_\alpha Q_\alpha + p_\beta Q_\beta)\right]$$
$$\times \exp\left[\varphi\left(\frac{\ln zq^{\pm1/4}}{2\pi\eta}\right) + \phi\left(\frac{\ln zq^{\mp1/4}}{2\pi\eta}\right)\right] :$$

where p_α, p_β are two arbitrary nonzero constants, γ_{E} is the Euler constant, and q and z are related to the parameters η and $\hbar$ via $q = \mathrm{e}^{2\pi\mathrm{i}\eta\hbar}$ and $z = \mathrm{e}^{2\pi\eta u}$.

Remark 3.3. We can set the parameter η in the above free boson representation to any fixed value without changing the representation itself. In this sense, the parameter η can be thought of as redundant. Indeed, the algebra $U_q(osp(2|2)^{(2)})$ contains only one deformation parameter, and its representation must necessarily contain no extra parameters.

Remark 3.4. From footnote 4, we see that the value of $\gamma = q^{1/2}$ corresponds to $c = 1/2$ in [9]. Thus the free boson representation given in proposition 3.2 is somehow not the most interesting one—remember that, for usual affine Lie (super)algebras, only representations at integer values of c have received attention, because only these representations are known to be unitary.

Remark 3.5. In [9], a free boson representation of the $U_q(osp(2|2)^{(2)})$ currents (8)–(14) at $\gamma = q$ (i.e. $c = 1$) was given. However, that representation does not have a well-defined limit as $q \to 1$ and hence the authors of that paper called their representation a 'nonclassical' one. One can verify that our representation does have a well-defined limit at this value of deformation parameter.

Although the representation of $U_q(osp(2|2)^{(2)})$ at $\gamma = q^{1/2}$ is not of the most interesting class, we could, however, use the same method to construct 'interesting' representations. Now, instead of (20)–(22), we introduce the following set of bosonic algebras:

$$[a_n, a_m] = -\frac{1}{n}(1 + (-q)^{-n})\delta_{n+m,0}$$
$$[b_n, b_m] = -\frac{1}{n}(1 - (-q)^{-n})\delta_{n+m,0}$$
$$[a_n, b_m] = [b_n, a_m] = \frac{1}{n}(q^n + q^{-n})\delta_{n+m,0}$$

together with the zero mode operators:

$$[P_a, Q_a] = 1$$
$$[P_b, Q_b] = 1$$
$$[P_a, Q_b] = [P_b, Q_a] = 0.$$

These bosonic commutation relations can also be realized in the tensor product of Fock spaces of two commuting sets of Heisenberg algebras, though we omit this here.

Defining

$$\varphi(z) = \sum_{n\neq 0} a_n z^{-n} + P_a \ln z + 2Q_a + 2Q_b$$
$$\phi(z) = \sum_{n\neq 0} b_n z^{-n} - (P_a + P_b)(\ln z + \mathrm{i}\pi/2) - 2(Q_a - Q_b)$$

we can easily prove

Proposition 3.6. *The following bosonic expressions give a free boson representation of* $U_q(osp(2|2)^{(2)})$ *currents at* $\gamma = q$*:*

$$E(z) =: \exp\varphi(z) :$$
$$F(z) =: \exp\phi(z) :$$
$$H^{\pm}(z) =: E(zq^{\pm 1/2})F(zq^{\mp 1/2}) : .$$

Again, this free boson representation is well defined as $q \to 1$.

4. Concluding remarks

The result in this paper provides an example of a two-parameter deformed quantum current algebra with the structure of an infinite Hopf family of (super)algebras and associated with a non-simply laced and twisted root system. To the authors' knowledge, this is the first example of this kind and hence a useful hint at the final classification of all such algebras.

Recent work on infinite Hopf families of (super)algebras has shown that very rich structures are incorporated within them. Though the relationship between such algebra families and the so-called quasi-Hopf algebras of Drinfeld [1] is not yet understood, positive progress is being made towards their Yang–Baxter realization [8], providing for the first time the possibility of applying these algebraic structures directly to solvable/integrable models in two dimensions (that is, without being forced to use Drinfeld's quasi-Hopf structure). More detailed work in this direction is currently under way.

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